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IN JUNIOR AND SENIOR HIGH SCHOOLS

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THE MATHEMATICS TEACHER

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EXPERIMENTAL COURSES IN SECONDARY SCHOOL MATHEMATICS

By RALEIGH SCHORLING
The Lincoln School of Teachers College

The National Committee on Mathematical Requirements, working under the auspices of the Mathematical Association of America, and numerous cooperating organizations, has prepared various reports dealing with the reorganization problems of secondary school mathematics. There is a growing sentiment against accepting materials of instruction or even outlines of courses that express merely an individual's opinion, especially if that opinion is not based upon class-room experience. But it is equally important to realize that a curriculum constructed in committee sessions, differs only in degree, not in kind. Fortunately, a milestone on the road of progress in education has been reached, which indicates that committee work that would eventuate, must be prefaced and supplemented by carefully made inventories of the experience that bears upon the problems studied. In consequence, a number of sub-committees have made studies of some of the details involved in the fundamental recommendations of the National Committee. One of these investigations deals with experimental courses in mathematics.

PURPOSE OF THE STUDY

The purpose of the study is to make generally available the experience of teachers conducting experimental courses in mathematics. Teachers will desire to know what is done by other teachers who are given greater freedom in choice of materials and methods, than is feasible in the typical public school. The value of this report is enhanced by the fact that the teachers in these schools, considered as a group, have wide experience, sound scholarship in mathematics, and unusual training in professional courses.

MATERIAL INCLUDED IN THE REPORT

The material includes a description of one or more courses in each of fifteen schools. Not all schools designated as experimental schools feel that they have made a contribution to the teaching of mathematics. The search for new materials by the National Committee on Mathematical Requirements has been wide spread. It is reasonably certain that no work considered unique by the persons doing the work has been omitted.

FORM OF THE REPORT

The whole report consisting of about 110 pages will be published sometime during the present year as a part of the final report of the National Committee on Mathematical Requirements. The report is presented in three parts—each for a special type of reader. The first part aims to meet the needs of the reader who is interested only in a general way, but who desires to survey the important implications of these innovations. These are presented in brief statements for the reader who does not desire to examine the evidence given in later parts of the report. The second part is for the type of reader who may wish to consider illustrations without going into the full details which constitute the basis for the statements in Part I. Part III is for the reader who has the time for and a genuine interest in considering more carefully some of the problems involved in improving the teaching of mathematics in secondary schools. This constitutes the source material of the report.

The material that follows is drawn from Part II of the report. It aims to amplify and to furnish illustrations of the important conclusions of the report.

1. *Only a Few Experimental Schools Have Made a Systematic Attack on the Mathematics Problem.* At the beginning of this study, a list of 175 schools which were giving courses more or less experimental in nature was prepared. It was desired at the outset to include every institution in which the teacher (or teachers) of mathematics felt that the experiment had progressed sufficiently for other teachers to profit by a detailed account. On this basis the list of experimental schools to be included was reduced to a brief one.

An experimental school must of necessity confine its constructive efforts to relatively few problems. Moreover, such schools are sometimes more concerned with becoming exponents of some philosophy of education or some educational ideal than in attacking a definite problem in a specific field. At any rate, there were very few experimental schools in which the teachers thought that they were teaching mathematics by other than conventional methods. Furthermore, it became clear that the report could not be complete without describing courses that are being taught in schools not generally considered experimental, as for example, Stuyvesant High School (New York City), East High School (Rochester, N. Y.), and the Cass Technical High School (Detroit, Mich.). The National Committee through its numerous cooperating organizations has solicited accounts of experiments in all sections of the country. The report includes a description of experimental courses in fifteen different schools.

II. *Experimental Work in Mathematics Not Scientific.* The word "experimental" is ordinarily used in a vague sense to indicate the use of materials or methods differing from those in current practice. But there is seldom an attempt to define the problem with precision or to measure the results; and rarely does anyone know when the experiment ends.

This leads one to say parenthetically that, viewed from an historical perspective, much of our accepted practice in high school mathematics has in this sense been experimental. The teaching of algebra alone in the ninth grade, and the teaching of demonstrative geometry in the tenth grade without propaedeutic training and the like, have at best been experiments in this sense, and perhaps very doubtful ones. In scientific education the word "experimental" has come to mean the study of a precisely defined problem under controlled conditions, and implies measurement of results. To have confined this report to such a definition would have excluded much material here presented which is nevertheless of great value. It represents at least the cumulative experience of critical, competent teachers. Accordingly, the reader will need to keep in mind that we are concerned in this report not so much with scientific experimentation as with purposeful innovations.

III. *Increased Emphasis Upon Experimentation.* We have reason to believe that the school world is putting a new emphasis upon investigation and controlled experimentation. It seems no longer sufficient to advocate an innovation on the basis of educational theory alone. Many school officials believe it highly desirable for a city or a school to incorporate experimentation as an important part of their program of education. The study of a problem by a group of teachers in a school system offers a means for stimulating discussion and genuine professional growth. S. A. Courtis includes controlled experimentation as one of the important duties of a supervisor. Such cities as Berkeley, Denver, Tulsa, Chicago, Rochester, Detroit, and many others, are engaged in setting up specific problems in mathematics and making it possible for groups of teachers to give them attention. To secure widespread cooperation in an experimental study that is clearly defined is no longer difficult. This increased interest in experimental work in mathematics is one of great significance and promise.

IV. *Teachers of Experimental Courses Well Trained.* The teachers of the schools described in this section have as a group an amount of professional training greatly exceeding that of mathematics teachers generally. Not only are they well prepared in mathematics, but also in general courses in education, educational psychology, measurements and special methods. The reasons are quite obvious. Some of these schools are in proximity to or directly connected with universities, thus affording opportunities for graduate study; in other schools the opportunity to do a constructive piece of work is the attractive feature; and in a few cases the salary is sufficient to command a choice of teachers.

The training of the teachers of the Cass Technical High School (Detroit) may serve as an illustration. In the year 1919-1920, the Cass Technical High School had thirteen full-time mathematics teachers. These teachers averaged 31½ semester hours preparation in college mathematics, and 13.8 semester hours in education courses. Roughly, this means that more than one-fourth of a four-year college course was devoted to mathematics, and a little more than one-ninth of a college course was spent in education courses. In addition, five teachers had completed

courses in mechanical drawing during vacant hours in regular day school classes, seven had taken machine shop courses to gain knowledge of machine operations and vocabulary, one had sufficient training in the industries so that he was considered one of the best machine shop instructors in the evening classes, and eight had been given temporary approval as teachers of trade improvement courses under the Smith-Hughes law.

Not only are these teachers well trained in mathematics, but they are especially trained for a special task. It is a well known fact that cooperation schools connected with large industries have great difficulty in securing teachers who are sufficiently well trained in education to do satisfactory teaching, and that in attempting to meet occupational needs the public schools find it difficult to obtain teachers who possess the necessary practical experience in the industries to use real problem material. It is safe to predict that the notion that a department head in an industry is a "boss" will be supplanted presently by the idea that he is first of all a teacher. And it is equally clear that the training for a teacher of mathematics in a technical school needs to give increasing emphasis to trade vocabulary, processes, and the real problem material. The Cass Technical High School suggests a solution to the problem, applicable alike to industry and the public school.

The averages given for the Cass Technical High School teachers are probably somewhere near the median for the group as a whole and serve to make clear the fact that this report is concerned with the work of teachers with very unusual training. It is important to keep this fact in mind in considering the principles concerning which there is substantial agreement in this group. The courses are unique not because they are taught by inadequately trained teachers recently released from college and restless for something new, but on the contrary, because they embody the mature judgment of experienced, well-trained teachers. These teachers constitute a group in which we may reasonably expect practice to approximate theory. This should be kept in mind in reading the issues stated in articles eight to fifteen inclusive of this part on which there is substantial agreement, and which are sufficiently far reaching to reshape completely the teaching of mathematics in American secondary schools.

V. *The Emphasis on Grades Seven, Eight and Nine.* Most of the efforts to reorganize mathematics are being directed at grades seven, eight and nine. The coming of the junior high school is only partly responsible for the especial attention that is now being fixed upon these grades. At the beginning of the seventh grade, the pupils may be expected to possess the fundamental skills in arithmetic. Hence, this is the time for new materials and an extension of number and space relations. This may explain why there appears to be greater variety in seventh grade courses throughout the country than in the fifth and sixth grade courses. The statistics of elimination and the experience with general intelligence tests show that we are rapidly becoming a ninth grade nation. Whatever the causes may be, the schools are beginning to hold the children through the ninth grade. The ratio of pupils entering the ninth grade has shifted in a decade from about one out of eleven to about one out of three. This fact may partially explain the unusual effort to recast ninth grade mathematics.

Grades seven, eight and nine are now available for inculcating common ideas, ideals, attitudes and habits. Hence the numerous efforts to build basic courses in these grades. All the schools in this report which include seventh grade classes teach courses that differ widely from conventional courses. On the other hand, these schools teach many senior high school courses that are frankly conventional. The early reports of the National Committee were particularly devoted to grades seven, eight and nine, and undoubtedly the discussion of these reports gives additional emphasis to these grades. It is, perhaps, logical to await the day for improving the later grades until the work of the earlier grades on which these rest shall have become more satisfactory.

It should not be inferred, however, that the senior high school field is wholly neglected. Presumably, the Ethical Culture* School's plane geometry course can be considered a senior high school project. Miss Blair's advanced course at the Horace Mann School for Girls is a marked innovation. Mr. Reeve at Minnesota and Mr. Breslich at Chicago have each developed

* The details of the courses referred to in this paragraph may be found in Part III of the complete report.

courses covering the senior high school. Miss Sanford of The Lincoln School teaches a course of college grade* to eleventh grade pupils, who have had but three hours of mathematics per week during the junior high school years. Both Mr. Schlauch's work (High School of Commerce) and the material of the Cass Technical High School touch the senior high school. It is of particular interest to note the efforts to teach the introductory notions of the calculus as is being done in at least three of the schools described, namely, The University High School (Chicago, Ill.), The Horace Mann School for Girls (New York), and The Lincoln School (New York).

VI. *No General Agreement Concerning the Details of the New Program.* The strongest evidence that much detailed and exacting work remains to be done in organizing a sound curriculum in mathematics is the fact that there is no general agreement concerning the details of the reorganized courses to be found in the practice of these teachers. In the content and in the grade placement of the materials in various junior high school books, we have clear evidence of distinctly different hypotheses for the organization of the curriculum of grades seven, eight and nine. Some schools, for example, believe that the seventh grade should be given entirely to arithmetic. One school gives about one-half of the year to arithmetic and the remainder to intuitive geometry. Several are working out courses in general mathematics for all three grades of the junior high school. The Ethical Culture School gives most of the ninth school year to geometry. There are almost as many kinds of courses as there are schools.

Disagreement concerning grade placement is to be expected, for very little careful work has been done on this problem. Nor have we attempted to measure to what extent the alternative schemes are effective in forming desirable habits, giving useful information and increasing mathematical power, the reason being that we do not now possess adequate tests. It is important to recognize that the issues involved in improving mathematics cannot be wholly met by a committee report or by wide spread discussion. As Professor Young has stated: "The work for

* In this course Young and Morgan's Elementary Mathematical Analysis is used. In the tenth grade the pupils study a conventional plane geometry text.

which the National Committee was organized truly begins when the recommendations of the report are being translated into practice by the detailed and exacting efforts of the classroom teacher."

VII. *Agreement Concerning Fundamental Issues.* It would be erroneous, however, to infer that all is chaos in the reorganization movement in mathematics. There is agreement concerning issues that are sufficiently far reaching to reconstruct completely our high school mathematics. This important evidence is to be found in the fact that the articles VIII to XV inclusive of Part I were submitted to each one of the schools concerned with the request that the teachers indicate their acceptance, modifications or rejection of these articles. All but one of the schools responded. (It was seemingly not possible to reach Professor Meriam.) With one exception, to be stated presently, there was unanimous endorsement and acceptance of principles. Mr. Betz, speaking for the Rochester teachers, takes exception to articles XI and XIII, not because of any objection in spirit but because he believes that courses are already being modified along these lines. The far-reaching effect of some of these principles will be noted in the sections that follow in this part.

Another striking illustration of agreement concerning important issues is found in an experience of the National Committee. It will be remembered that the Committee published in Secondary School Circular No. 6, page 10, five suggestions for the arrangement of material for grades seven, eight and nine. Each one of these plans provides that at some time or other during grades seven, eight and nine, the pupils shall be taught applications of arithmetic, algebra, geometry and trigonometry. But a more startling fact is that in no one of these plans is it proposed that the whole year be given to any one of these subjects (arithmetic, algebra, or geometry). According to the plans proposed, there are but two alternatives: (1) the teaching of general mathematics throughout these years, or (2) the breaking up of each year into various units, algebra, intuitive geometry, trigonometry, and the like. Is it not remarkable that, after a wide spread discussion of junior high school courses involving nearly one hundred cooperating organizations and in spite of the fact that prevailing current practice devotes all of the sev-

enth and eighth years to arithmetic and all of the ninth year to algebra, there should not be enough sentiment in the country to support at least one plan for the junior high school that would continue conventional practice?

VIII. *The Norms of Performance in Arithmetic Are Exceedingly Low.* Teachers of mathematics have in recent years had an opportunity to study the findings of numerous school surveys. One outstanding conclusion to be drawn is that the results obtained in the arithmetic of grades seven and eight are very unsatisfactory. We assume that in the first six grades the child shall attain a reasonable degree of accuracy and speed in the fundamental operations with integers and fractions. The result of tests show that this assumption is not even realized in the seventh and eighth grades. We have sufficient evidence to make us sympathetic with the complaints of the industrial and commercial world resulting from its contact with pupils who have achieved only 50% to 80% accuracy. This failure is probably not due to an insufficient amount of drill; it seems to be due, rather, to the fact that the drill is not directed at specific disabilities. The results of available tests do not in general enable us to analyze the pupils' dificulties. For example, one may give the Courtis Test in subtraction and learn that a pupil or a class is weak in subtraction. But the tests do not tell us what steps in the learning are in need of remedial treatment.

The foregoing conditions seem (without evidence based on standard tests) to exist with reference to percentage relations. This topic is generally considered important and it receives emphasis in text books. But the degree of mastery of percentage relations seems low even among college students. The experience of the teachers in these schools suggests that a satisfactory skill in computation cannot be assumed as initial equipment even among ninth grade pupils, and that adequate drill must be provided from time to time. In short, computation must be "kept going."

IX. *Delay of Social-Economic Arithmetic.* Although there is an increasing emphasis upon material that can be defended on a basis of social usage, it does not follow that a problem which involves principles that are widely used in every day affairs is either a real or desirable problem for children. Much emphasis

is ordinarily placed on mortgages, stocks, promissory notes, cost of plastering, papering, and the like. These may at times be of interest to parents and to teachers, but it does not follow that these problems are vital to all the pupils or to any considerable portion of pupils in a class of twelve to fourteen-year-old boys and girls. There is a marked decrease in the amount of textual space devoted to these social-economic materials to be found in all eight of the junior high school seventh grade books that have as yet appeared. It is generally believed by teachers of these experimental courses (with the possible exception of the Trenton group) that this type of material calls for more maturity than is ordinarily possessed by seventh grade pupils. This does not mean that time should not be devoted to the arithmetic of the home and to a growing familiarity with common business forms, checks, personal accounts, and the like. But it is held that the applications of arithmetic to business may well be decreased in the early years and continued later in grades eight and nine so as to capitalize the increasing maturity, social experience and mathematical power. If we are getting to be a "ninth grade nation," the proposed suggestion is not only desirable but feasible. The delay of this material to later grades is advocated because of the demand of modern secondary education that social needs be more adequately met. These materials should be taught at a time when they are of the greatest profit to the pupils—keeping in mind that they must be taught before the pupils leave school.

X. *The Appeal of Mathematics.* There is a desire on the part of the teachers of these experimental courses to construct courses in mathematics that shall make a direct appeal to the pupil as being worth while here and now. If the new courses in mathematics are to respond to the modern demands in education, they must "reveal higher activities and make them both desired and to an extent possible." This suggests that there is to be less pyramiding (the practice of including elements to be used in problems occurring in later courses), and a more direct attack on the chief objective: an increased understanding and appreciation of mathematics. It is claimed that there are schools in this country giving rigorous courses in mathematics where the question "Why study mathematics?" or

"Why study this topic" is no longer asked in the spirit of dissatisfaction. Much thought is being given in these experimental courses as to how this number may be increased.

XI. *Formalism of Ninth Grade Algebra.* Extreme emphasis upon the manipulation of symbolism which does not have meaning and significance to the pupil, together with poorly selected content and material of little value even in later courses of mathematics, constitute the two pronounced indictments of the standard algebra of the ninth grade. This indictment has been often expressed, even by conservative mathematicians.

The standard ninth grade course in elementary algebra is in theory and symbolism substantially what it was in the seventeenth century. Fully 70% has been shown to be mere manipulation of symbols and drill on non-essentials set up fifty years ago. Ninth grade algebra has remained static, while the quantitative needs of high school children have changed. To be sure, we no longer teach as many as seventeen cases of factoring. But the Brinton study* makes it reasonably certain that ordinarily schools spend from four to six weeks on special products and factoring, three weeks on simplifying fractions, and something like eighteen more weeks of the school year on material equally useless. Experienced teachers and psychologists alike agree that pupils can learn to do this manipulation without associating it with meaning. Much of the manipulation of symbols in the ninth grade leads nowhere, not even in later mathematics. Professor Crathorne, who has had the opportunity each year to inventory the mathematical equipment of hundreds of Illinois freshmen, asserts that many of these students can factor with more facility than their instructors, and yet have but little understanding or appreciation of the simplest mathematical principles. It is significant that every one of the schools herein represented is concerned with the problem of simplifying the algebra of the ninth grade. Indeed the Las Vegas School went as far as to eliminate for a time all mathematics from the ninth school year in order to avoid objections to ninth grade algebra.

XII. *The Exclusiveness of Ninth Grade Algebra.* Not only are the teachers of these experimental courses concerned with

* See Lee Brinton's question blank study, Relative Importance of the Topics of Algebra reported in Supplementary Educational Monographs, Vol. II, No 1. (The University of Chicago Press.)

the formalism of ninth grade algebra but they are determined to break up its exclusiveness. We have noted that the teachers of the Las Vegas School began their experiment by teaching no mathematics at all in the ninth grade. They chose to do this rather than to teach useless material. But they have come to believe that it is more important to strive to develop the right kind of course for the ninth grade. The exclusiveness of algebra has delayed to later courses the teaching material that is simple, worth while and significant to children. The most patent illustration of delayed material is found in numerical trigonometry. In the high schools of our large cities, most teachers have sufficient preparation to teach trigonometry, or at any rate some one in the department has. But even the simplest notions in this subject are seldom referred to until we reach the eleventh and more frequently the twelfth school year. This, in spite of the fact that the introductory notions of trigonometry, covering possibly three or four weeks' work, are exceedingly simple and are often the means of awakening a genuine and abiding interest in mathematics. While the case of trigonometry is more familiar to teachers of mathematics, other topics are perhaps even more important and more extensive. The graph is seldom introduced early or, if included, is not consistently taught as a vivid method for solving problems. The arithmetic involved in the elementary notions of statistics, the simple principles of mechanics, practical devices such as logarithms, the slide rule, and the like are equally neglected. So seldom are these topics taught that most high school graduates are denied the opportunity to obtain even elementary notions of these important tools. It is not necessary to argue the early introduction of these topics on the basis of practical usefulness. It is sufficient to point out that these constitute most effective ways of presenting quantitative data and relationships to children.

Presumably the aims of reorganizing the mathematics of the ninth school year are identical with those for grades seven and eight. These aims seem to be well formulated in a report by the Chicago Principals Association published in full in the September number of the *Chicago Schools Journal*. According to this report the aims of teaching mathematics in these early years are: (1) "to apply computational arithmetic to the most useful

phases of business practice and social applications, (2) to study space forms and relationships arising in linear, surface, volume, and angular measurements, (3) to study and to represent quantitative data and relationships by means of simple graphs, (4) to express quantitative relationships in the language of the formula and the equation and to evaluate and solve the same, (5) to study positive literal numbers and the necessary forms and operations to evaluate simple formulas and to solve simple equations, and (6) to practice at times on the fundamentals of computation until a norm of proficiency is acquired."

In the discussion of Section VII it has already been pointed out that the five plans proposed on page 10 of Secondary School Circular No. 6 nowhere provide for a year to be devoted exclusively to algebra. The suggestion is that American Secondary Schools have experimented with the exclusiveness of algebra in the ninth year and have pronounced it a failure.

XIII. *The Special Difficulties of Demonstrative Geometry.* Teachers of mathematics have not surrendered the contention that plane geometry offers possibilities for practice in rigorous thinking. It is still believed that this field is rich in possibilities for giving a method of attack in problem solving and for analysis and, especially, for inculcating ideals as to precision and logical thought processes. It is also quite clear that the time and money spent on plane geometry cannot be justified on the basis of social utility. It must be defended for its contribution to a systematic technique of thinking or it must be omitted. In so far as conclusions are valid from questionnaire studies, it must be admitted that men and women of affairs are still guided by the conviction that plane geometry offers much in addition to information.

One may accept this argument presented by both groups as correct and yet insist that plane geometry as taught is an intolerable failure, because it falls far short of its possibilities. In his presidential address Professor Moore* pointed out the unwisdom of forcing children prematurely into this type of thinking. To be sure, a certain number in a geometry class who are chronologically quite young may succeed in doing plane geom-

* Science, March 13, 1903.

etry well. But this only furnishes additional evidence for Professor Moore's contention, for these children invariably have, as far as known, high intelligence quotients, and are therefore of high mental age.

Recognition of the special difficulties of plane geometry is clearly shown in the material of Part III of this report. To be sure, there is a very wide difference of opinion as to how the problem is to be solved. The Ethical Culture School, which perhaps has devoted most attention to plane geometry, offers one solution. The University of Chicago High School offers one that differs markedly. But the various plans are characterized by this important agreement that pupils shall no longer be permitted, much less required, to take a plane geometry course in the tenth or later grades wholly unprepared. If geometry is to be required, we can at least give pupils with fair (or greater) ability a chance not only to pass the course, but also to get a considerable amount of satisfaction from it. In many of these schools, plane geometry appears to be the most popular course, even with some pupils whose intelligence quotients are as low as "95-105." In every case, the pupils have been given the vocabulary, the symbolism and a preliminary organization (achieved through intuition and experiment) in the years that precede the plane geometry course. In some of the schools this preparatory geometry is given as a separate unit. For example, the Horace Mann School for Girls offers a plane geometry unit in the last half of the seventh school year. In The Lincoln School this geometry takes the form of a study of space forms and relationships arising in linear, surface, volume and angular measurements, and in the representation of quantitative data and relationships by means of simple graphs. In the measurement work of this school, it forms a basis for building up the concept of an algebraic number. It leads at once to the formula and the equation, to the evaluation of algebraic numbers, and to the substitution of special values in the formula. In this school the geometry is continued throughout the three years that precede demonstrative geometry. Most of the schools herein described are found somewhere between these two practices. The purposes in these various forms are identical, namely, to give the pupil the usable values much earlier, and to increase greatly

the probability that he can later pursue a course in plane geometry with interest and profit.

XIV. *Content in Grades Seven, Eight and Nine.* There is agreement that the work of grades seven, eight and nine should include arithmetic, intuitive geometry, algebra and numerical trigonometry. Although some schools advocate the addition also of a unit of demonstrative geometry to the ninth school year, this issue is still debatable. In the Horace Mann School for Girls will be found a provision for a half year unit of demonstrative geometry. The teachers in that school argue that the junior high school years must give all an opportunity to know what is meant by a *proof*. The Ethical Culture School gives a more extensive unit of plane geometry in the ninth year. In the work of the University High School of Chicago, there is an attempt to have a gradual transition from the intuitive to the demonstrative. At no time, however, does the student study plane demonstrative geometry as such.

We see from the foregoing that the tendency is to make the work of these three grades more inclusive. Still further evidence is found in the tendency to introduce earlier than is now customary certain topics and processes which are more closely related to modern needs, such as the use of logarithms and other simple tables, the use of the slide rule, simple work in arithmetic and geometric progressions, the elementary ideas concerning statistics, the technique of graphic representation as a tool in problem solving, and the like. While these topics and others were listed in a preliminary report of the National Committee (Secondary School Circular No. 5) as optional, their appearance in so important a report dealing with the work of the ninth and tenth school years shows the tendency of the times. More material concerning these topics may be found in Part III of this report in the sections dealing with the Stuyvesant High School, The Lincoln School, the High School of Commerce, the English High School (Boston), and others.

XV. *The Bearing of College Entrance Requirements on the Early Years of the High School.* The mistaken notion that it is necessary to formalize the teaching of mathematics throughout the years of the high school in order to make a satisfactory showing on college entrance examinations needs to be overthrown.

There is sufficient evidence in the reports that follow in Part III to show conclusively that it is possible to give worth while courses in the early years of the high school with little reference to college entrance examinations. The reader will learn that the graduates of the English High School (Boston) do creditable work at the Massachusetts Institute of Technology. The graduates of the University High School of Chicago leave little to be desired in their subsequent record in mathematics. The Lincoln School (New York) devotes but three hours per week to mathematics in grades seven, eight and nine; five in grades ten and eleven; and one hour per week (review work) in the twelfth grade. No special lessons have ever been given, nor has tutoring been permitted. The graduates of this school not only are able to pass college entrance examinations, but are given advance credit at colleges of the first rank. The Cass Technical High School (Detroit) has an excellent record at the University of Michigan, to which it sends many of its pupils to take later courses in mathematics.

It is possible that the indictment that college entrance examinations are not the stimulus toward valuable work in mathematics, which they might well be, can be substantiated. But bad as they may be, they are not the obstacles to worth while courses in mathematics that they are frequently believed to be.

THE NATURE OF ALGEBRAIC ABILITIES*

(Continued)

By EDWARD L. THORNDIKE
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Ability with Problems

One of the most revered features of algebra is training in organizing a set of facts given in a problem described in words, into an equation or set of equations such that solving will produce the desired answer. This, so far as the problems are genuine ones whose answers a sane person in the real world might seek, is admirable. The genuine problems appropriate to a reasonable life do not, however, often lead to such fractional expressions, or quadratic equations, or denominators that need to be rationalized, or sequences wherein x appears three or four times, as are being studied and await "application to problems." The genuine problems are mostly of a type where x appears once, the other elements being numbers, and require only straightforward arithmetic plus certain conventions with respect to parentheses. Consequently, problems have been made up to give the pupil trainning in applying his more subtle algebraic techniques. These pervade the textbooks, courses of study and examinations. The following are samples:

> The earth and a certain number of planets revolve around the sun. Twice the number of planets which are nearer to the sun than the earth is plus one equals the number of planets which are farther from the sun than the earth is. Find the number of planets nearer to and the number more remote from the sun than the earth is.

> If a railroad train is made up of five sleeping cars, one parlor car and a certain sort of engine, its cost is $129,200. The cost of each sleeping car is $300 more than the cost of the engine. The cost of the parlor car is five-sixths of the remainder when the value of the engine is diminished by $100. Find the cost of the engine, the parlor car, and of a sleeping car.

> The front wheel of a cart makes 16 revolutions more than the rear wheel in going 360 feet. If, however, the circumference of the front wheel were increased by a third, and that of the rear wheel by a fifth, the front wheel would make only 10 more revolutions than the rear wheel in going the same distance. Find the circumference of each wheel.

* The studies reported in this article were made possible by a grant from the Commonwealth Fund.

The hours required by Mr. A to travel a certain distance equals the number of miles he travels per hour. Mr. B goes the same distance in 2 hours less time by going three miles more per hour. Find the rate of travel and time taken by Mr. A.

There are two angles, one of which is 5° less than the other. If the number of degrees in each is multiplied by the number in its supplement, the product obtained from the larger of the given angles exceeds the other product by the square of the number of degrees in the smaller of the given angles. Find the angles.

Shall the ability to solve problems mean the ability to solve such as these, or shall it mean the ability to solve genuine problems of a sane life?

It has been customary to select and arrange the problem materials almost wholly from the point of view of the algebraic technique to be applied. The teacher or textbook maker, having taught the pupil how to operate with algebraic fractions, for example, looks about for problems which will lead to fractional equations. Other characteristics are treated as of minor importance. From the functional point of view, emphasizing ability to use algebra in solving problems which life will offer, it seems desirable to consider the lives of boys and girls and men and women as students, citizens, fathers and mothers, lawyers, doctors, business men or nurses, and select problems which they may usefully solve and which are properly solved by algebraic methods. These may then be arranged according to the technique involved if this is desirable; but from the functional point of view much is to be said in favor of arranging them with consideration also for their natural connections in the world of fact and their logical connections in the mind. Problems about public health, for example, may well be on the same page even though one should involve a simple equation without fractions, one a fractional equation, one a radical and one a quadratic.

The arrangement is of much less consequence than the choice. Most important of all is the general choice of attitude—are problems to be an exercise ground for algebraic insight, or is algebra to be a tool for life's needs?

The quantitative problems of life usually come in connection with real things, events and relations. There are real fields and floors, or at least maps and house plans; real ships to be navigated, guns to be pointed, alloys to be compounded, medicines to be diluted, electric cells to be connected. The abilities eventually desired are preferably abilities to deal with real situations.

Since it is inconvenient to provide these real situations in schools, we have recourse to verbal descriptions of them. We cannot, however, take it for granted that the ability to manage a certain quantitative problem as it is described in words is identical with the ability to manage the same problem when it actually arises in a real situation. The difficulties of the described problem may be largely linguistic; to take an extreme case, a person obviously could not solve a problem no matter how easy it was in reality, if it were put to him in an unknown language. On the other hand, the verbal description may be far more suggestive of the procedure to follow than the real situation would be. If a boy should think "In how many years shall I be half as old as my father?" and proceed to solve the problem he would have to know enough to ask, "How old is he now?" "How old am I now?"; and he might puzzle about exact birthdays, even the time of day of birth or allowance for leap years. In the described problem, "A boy is 14. His father is 40. In how many years will he be half as old as his father?" he is given all the data needed, and encouraged not to bother about anything more than getting a certain number which, if added to 40 and 14, makes one result twice the other.

In proportion as we retain the older view that the main value of problem-solving is its formal disciplinary value as a mental gymnastic, the distinction between ability with a problem as it occurs in reality and ability with a similar problem described in words approaches zero importance. We may even deliberately make the verbal description much easier or much harder to understand than the corresponding real situation would be.

This view is, however, hardly tenable with respect to problems that assume to have anything to do with real situations of business, science, technology, or the home. If problems have only formal disciplinary value as mental gymnastic, we may as well use unreal problems about the square of somebody's age being equal to the cube of half of his age less $2\frac{1}{2}$ times his age, or about consecutive numbers, or about fractions whose numerators and denominators are related in divers ways. If we have problems about realities at all, we probably have them to train the pupil to manage realities themselves rather than to manage words about them.

One special difference between problems arising in connection with situations actually present to sense and the customary verbal problems of the algebra class, is that in the former needed data may be missing and irrelevant data may be present in large numbers, whereas in the latter we have practically accepted it as a rule that every problem should be solvable from the data given without any further additional data and that all the data given in the problem must be used in order to attain the solution.

There may be in use in some schools problems where the pupil has to decide whether the data are adequate for its solution and problems where he searches elsewhere for the additional data needed, but they are surely very rare. Problems where the data to be used in framing the equations have to be chosen from amongst many data irrelevant for that particular problem are almost equally rare.

Our custom in schools of making each problem a little paragraph of statements all to be used without adding or subtracting one jot or tittle, like the pieces in a picture puzzle, is so fixed that we find one of our standard textbooks announcing the following:

> 1. Every problem gives a relation between some unknown numbers.
> 2. There are as many distinct statements as there are unknown numbers.
> 3. Represent one of the unknown numbers by a letter; then, using all but one of the statements, represent the other unknowns in terms of that same letter.
> 4. Using the remaining statement, form an equation.
>
> (Wells and Hart, '12, p. 100.)

Is problem solving in algebra to be only a puzzle game of fitting translated phrases into a proper equation?

There is, then, a wide range of possible opinion about what problem solving does mean and about what it should mean. We should surely try to make it mean something more educative than the solution of more or less attractive puzzles made up to exercise algebraic technique or to give indiscriminate practice in "thinking." We shall return to this subject in a separate article where the attempt will be made to clear up the psychology and pedagogy of problem solving in algebra.

The psychological demands and the psychological effects of organizing the facts of a problem into an equation or equations such as will, when solved, give the desired answer varies greatly

according as the problem is, so to speak, an "original" which the pupil thinks out, or follows other similar problems which he has learned to handle by special training *ad hoc*.

The approved theory has been, and still is, that the former should be the process in the main, but skilful teachers seem to think or fear that actual proficiency in solving the problems that are met in life (or at least those that are met in examinations) is best secured by special training in certain routines such as "let x equal the smaller number, when there is a choice," or "be sure to use all the numbers that are given," and by still more specialized training with rate problems, mixture problems, tank-and-pipe problems, and the like.

Rugg and Clark argue that directed practice with many different kinds of problems will make the pupil "able to use the method in solving any kind that you may happen to meet later" ('18, p. 208); and provide this directed practice with (I) Problems relating to age, (II) Problems in which a number is divided into two or more parts, (III) Problems based on coins, (IV) Problems based on relations between time, rate, and distance, (V) Problems involving percents, (VI) Problems concerning perimeters and areas, and, (VII) Problems based on levers. Their treatment of the first group is as follows:

"Section 97. *Need for tabulating the data of word problems.* Many problems involve so many different statements that it is practically *necessary to arrange the steps in the translation in very systematic tabular form.* Take an example like this:

" 'John's age exceeds James' by 20 years. In 15 years he will be twice as old as James. Find the age of each now.'

"Before we can write this statement in the form of an equation we must express in algebraic form *four* different things: (1) John's age *now;* (2) James' age *now:* (3) John's age in 15 years; and (4) James' age in 15 years. These four facts can best be stated in a table like this:

"(First step) Let n represent James' age *now*.

"(Second step) Tabulate the data.

TABLE 15

	Age Now	Age in 15 Years
John's age	$n + 20$	$n + 20 + 15$
James' age	n	$n + 15$

"With all the facts expressed in letters we can now state the equation which *tells the same thing as the original word* statement; namely:

"(Third step) $n + 20 + 15 = 2(n + 15)$

"We are now ready for the

"(Fourth step) the solution of the equation; the steps are as follows:"

Then follow explanations of solving and checking and sixteen carefully graded tasks leading up to five of the customary age problems such as, "A man is now 45 years old and his son is 15. In how many years will he be twice as old as his son?" ('18, p. 208.)

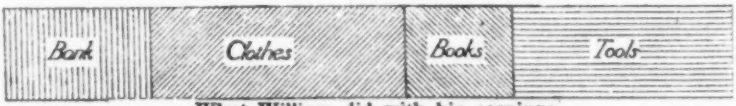
Shall we treat the problem material as a series of originals, or as a collection of typical groups of problems, each group of which the pupil learns how to solve much as he learns how to subtract a negative number or multiply x^a by x^b by adding exponents? Or shall we treat part of the problem material in the one way and the rest in the other? This last is just what we do in arithmetic. Certain groups of problems (as about areas, perimeters, discounts, insurance, compound interest, taxes and commissions) are prepared for by special training with each. Certain other problems are left to the undirected ingenuity of the pupils.

Ability with Graphs

The ability to understand, construct and use graphs needs definition in at least two respects. (A) Shall school algebra present elementary facts concerning all graphs that are simple and important, or shall it limit itself to the graphic presentation of a relation between two variables through the Cartesian coordinate system, and to such introductory matter as facilitates that? (B) Assuming the second answer to A, shall it deal with graphs of irregular relations not presentable in any equations which the pupil can be expected to manage, or shall it restrict itself to the straight line, parabola, hyperbola, and the like? Question A may be made clear and vivid in this form: "Which of the types of graphs on pages 85 and 86 shall the pupil be taught to understand, construct and use?"

The graphs which are simple and important may be classified (I) descriptions of the way in which a certain quantity is

divided; (II) general comparisons of two or more magnitudes; (III) comparisons of two or more magnitudes which are put in order by their relation to some characteristic; and (III-A) comparisons of two or more magnitudes which are themselves frequencies of occurrences and are put in order by their relation to the magnitude of some characteristic.



What William did with his earnings

What Louise did with her earnings

FIGURE 1

TYPE I. DIVISION OF A QUANTITY.

1. These diagrams show what William and Louise did with the money which they earned last summer. Make a table showing what each of them did with the money. What percent of his money did William put in the bank?

2. What percent did he spend for clothes? For books? For tools?

3. What percent did Louise spend for clothes? For books? For music?

TYPE II. COMPARISON OF MAGNITUDES.

1. According to the diagram how many pounds can Charles lift? How many can Dick lift? Fred? Tom?

2. Draw a diagram to show how many of the exercises in the arithmetic practice on page 165 each of these children did correctly, in 5 min.

Alice, who had 30 correct
Anna, who had 19 correct
Nell, who had 24 correct
Sarah, who had 36 correct

Let ⅛ in. of distance up and down equal 1 exercise correct, 1 in. equal 8 exercises correct, etc.

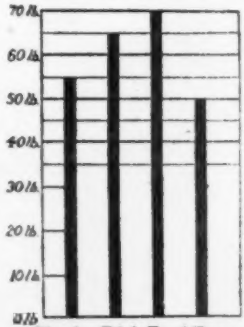
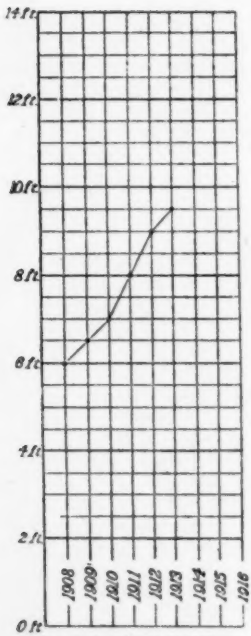


FIGURE 2



Records of Dick Allen in the broad jump

FIGURE 3

TYPE III.

COMPARISON OF ORDERED MAGNITUDES

3. Using thin paper trace and complete this diagram or graph, which tells how Dick Allen improved in the broad jump. His records were 8 ft. in 1911, 9 ft. in 1912, 9 ft. 6 in. in 1913, 10 ft. 6 in. in 1914, 13 ft. in 1915, and 13 ft. 6 in. in 1916.

4. Draw a diagram or graph to show how Elsie improved in repeated trials with the practice test on page 165.

Her scores in the 10 successive weeks were 17, 17, 19, 23, 22, 23, 24, 25, 24, 26.

5. Do the practice of page 165 twice a day for five days. Draw a graph showing how well you did in each trial and how much you improved.

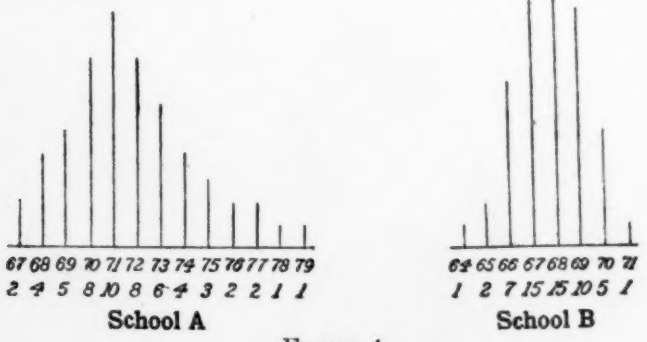


FIGURE 4

TYPE III-A. GRAPHS OF RELATIVE FREQUENCIES.

5. Examine the diagrams on p. 196. Read the diagram for School A, saying, "On 2 days, the temperature was 67°; on 4 days it was 68°; on 5 days it was 69°"; etc. Find what percent of the days were "satisfactory" as to temperature in School A. In School B. (A temperature of 66, 67, 68, 69 or 70 is called "satisfactory.")

Figure 1 is an illustration of graphic descriptions of the way in which a certain quantity is divided, as when we wish to show how a family spends its income, or how the population of a country is divided with respect to race or how a pupil spends the day.

Figure 2 is an illustration of general comparisons of magnitudes, as of the number of boys and of girls in a school, or of the values of certain products of a farm, or of the number of voters of each political affiliation, or of the size of a school ten years ago and now, or the percent of illiterary in each of forty states.

When the magnitudes to be compared form a series easy to put in order by their relation to some characteristic, this is usually done. If, for example, we have the size of a school now, five years ago, ten years ago, fifteen years ago, etc., it is obviously best to put the columns or bars showing size in a chronological order. If we have the number of children at each age in a school, it is obviously best to put the columns or bars in the order: aged 5, aged 6, aged 7, etc.

Figures 3 and 4 illustrate such graphs of class III. The typical graph of the relation of one variable to another is the most notable case of class III. Graphs of surfaces of frequency or distribution (III-A) are a group of increasing importance in the social sciences.

It is customary to draw a distinction between statistical graphs and mathematical graphs. The former include all of classes I and II and those of class III which are not readily analyzable into regular relations conveniently expressed in the relation of y to x. This distinction is not sharp or rigid, some weight being also given to custom and convenience.

Return now to our question A, "Shall school algebra deal with graphs of classes I and II, or shall it limit itself to class III?" The future will probably save us the trouble of answering, because acquaintance with graphs of classes I and II, and with simple cases in class III, will probably be given in grades 4 to 8. Figures 1, 2, 3, and 4 are, in fact, all from a recent textbook in arithmetic. When it has not been given already, such work may deserve a place in ninth grade mathematics because

of its intrinsic worth, and because of the interest it lends to the graphic presentation of the relations between two variables.

Graphs of class I and class II are, however, psychologically useful in algebra only as possible introductions to those of class III. For pupils at the high school level, the erection of a series of columns and the formation of the curve joining the mid-points of their tops, seems adequate. Also it seems to the writer that the development of a serial graph from the graph of mere comparison will introduce more interferences than aids to comprehension of the former. Logically, the graph of a systematic, ordered relation may be thought of as a sub-class of, and development from, the graph comparing any quantities—the population of states, or the scores of ball players, or the heights of species of trees. But it seems sounder psychology to teach the systematic graph, ordered in relation to time, or number of articles, or size of the object, or force exerted, etc., by itself.*

The elementary facts of surfaces of frequency deserve a place in the curriculum somewhere in grades 6 to 9 (preferably in grades 7 or 8), the pupil being taught to understand such graphs and to construct them from the tabular data. Their relation to the curve $y = e^{-x^2}$, and to the coefficients of the binomial expansion may perhaps be shown toward the end of a course in algebra.† Their consideration along with the graphs of simpler relations of one variable to another will be confusing and should be avoided.

Our second question was: Assuming that algebra in grades 9 and 10 limits itself to "mathematical" graphs of the relation of one variable to another and to such introductory matter as is of value therewith, shall it deal with graphs not presentable in any equations that the pupil can master, or shall it restrict itself to the straight line, rectangular hyperbola, parabola, circle, and the like?" It may be answered provisionally as follows: Enough work with irregular relations should be given so that the pupil will appreciate the regularity of regular ones by contrast, and also the place of these regular relation lines amongst relation lines in general. Except for that, it seems best to spend the time on standard forms whose mathematical significance can

* That is, to make the principle of organization the facts and laws to be expressed, rather than the general nature of graphic presentations.
† More suitably, probably, in an advanced course.

be realized.* The curves for $y = ax$, $y = ax + b$, $y = x^2$, $y = x^2 + a$, $y = x^3$, $y = x^3 + a$, $y = \frac{a}{x}$, $y = x^2 + ax + b$, $y^2 + x^2 = a^2$, and $y = a^x$ will probably provide sufficient variety to make the principles general.

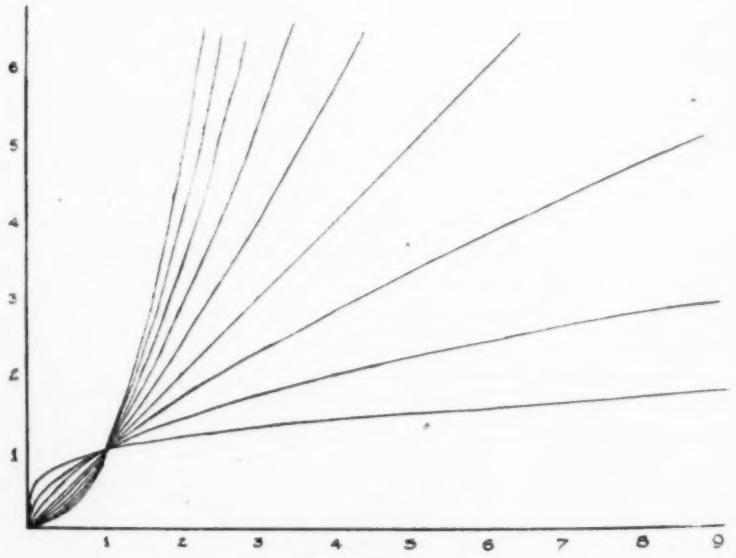


FIGURE 5

When the topic of fractional exponents is taken up, a review and extension of the graphic treatment of regular relations may be had by presenting the curves for $y = x^{\frac{1}{2}}$, $y = x^{\frac{2}{3}}$, $y = x^{\frac{3}{4}}$ $y = x^{\frac{3}{2}}$, $y = x^{\frac{4}{3}}$, and the like, as in Figure 5. If logarithms are taught, the construction of such curves provides excellent exercises in computation and approximations, and a useful application of knowledge of fractional exponents. Simple facts about $y = k^x$ may be taught also.

* Including, perhaps, standard forms obscured by chance errors which disappear in a coarser grouping of the data.

THE VOLUME OF A SPHERE*

Granted—If a moving plane be constantly perpendicular to a fixed line which passes through a given point of the plane, any two figures in the plane which are equal in area will generate equal volumes.

Corollary—If the two figures in the plane are not equal, the two volumes generated will be to each other as their generating areas in the plane.

Theorem—The volume of a sphere is equal to one-sixth of a cube upon its diameter multiplied by the circummetric ratio $\cdot \pi$.

In the figure, let ABCD — E be a sphere, AC its diameter, V its volume, and π the circummetric ratio.

$$\text{Then will} \quad V = \frac{\pi}{6} \overline{AC^3}$$

Let ACHG be a rectangle whose plane cuts the sphere in the great circle ABCD. In this plane produce $\overline{CH}$ to K, taking HK = HG, and draw GK; also, through G, perpendicular to the plane GHK, draw GL = GH, and draw LH and LK. Then will GHK and HGL be two right triangles, right angled at H and G, respectively and, having their planes perpendicular to each other.

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* Some years ago Prof. J. B. Johnson turned over to me a trunk fairly well filled with papers and manuscripts originally belonging to and largely written by Alonzo Jackman, Adjutant General of Vermont during the Civil War and a professor at Norwich University from 1839 till he died, during the Winter of 1879. (During the last five of those years Johnson and I were in his classes in mathematics.) Among the manuscripts were: a trigonometry, a treatise on conic sections, a detailed description of an astronomical telescope with revolving mercurial reflective very much like the one invented nearly thirty years later; and this demonstration. I had not the time then to examine all the papers with care; but I had copies made of some, including the "Volume of a Sphere," which follows essentially as written, save the omission of certain references to texts long ago discarded if not forgotten.

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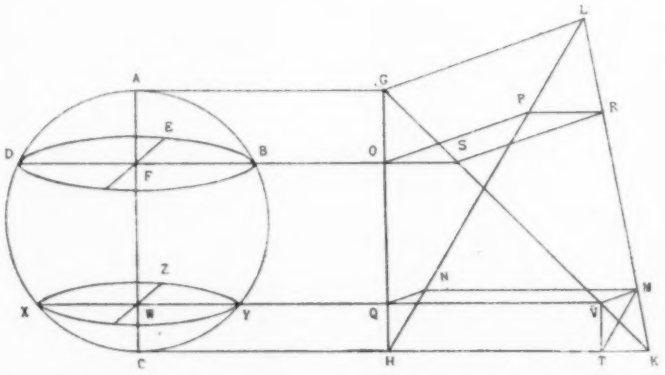
The planes GHK, GHL, KHL and KGL form a pryamid whose base is GHK, whose altitude is $\overline{GL}$, and whose volume $\frac{1}{2}\overline{GH} \cdot \overline{HK} \cdot \frac{1}{3}\overline{GL}$. Or, since $\overline{GH} = \overline{HK} = \overline{GL} = \overline{AC}$ by supposition, the pyramidal volume, $GHK - L = \frac{1}{6}\overline{AC^3}$.



Through any point of $\overline{AC}$, as F, draw a plane perpendicular to AC, or GH, and denote the sections it cuts in the sphere and the pyramid by DEB and OPRS. GL and HK will be parallel to this plane, and DEB will be a circle whose center is F.

Since $\overline{OP}$ and $\overline{SR}$ are both parallel to $\overline{GL}$ they are parallel to each other; and for the same reason $\overline{OS}$ and $\overline{PR}$, parallel to $\overline{HK}$, are parallel to each other. Therefore, OPRS is a rectangle, right-angled at O. Further, since $\overline{GH} = \overline{HK} = \overline{GL}$, we have in the triangles GHK and HGL, $\overline{GO} = \overline{OS}$ and $\overline{HO} = \overline{OP}$, whence $\overline{GO} \cdot \overline{OH} = \overline{OS} \cdot \overline{OP}$ = area OPRS. But $\overline{GO} \cdot \overline{OH} = \overline{AF} \cdot \overline{FC} = \overline{FD^2}$. Consequently the area OPRS $= \overline{FD^2}$.

Now $\pi\ \overline{FD^2}$ equals the area of the circular section DEB, and, therefore, the section DEB $= \pi \cdot$ OPRS.

Since the same ratio will exist between any two corresponding sections of the sphere and the pyramid wherever the plane

cuts AC, if we conceive the plane to move continuously through all points from C to A carrying with it these corresponding sections, one of them will generate the sphere and the other the pyramid, and the sphere will be π times the volume of the pyramid; or

$$V = \frac{\pi}{6} \cdot \overline{AC}^3$$

COROLLARY

Any segment of the sphere cut off by a plane perpendicular to AC will have a volume equal to π times the corresponding portion of the pyramid—or, in the figure, the segment $XYZ - C = \pi \cdot QNMV - KH$.

Pass a plane through MV parallel to QNH cutting HK at T.

The volume of the pyramid $MVT - K = \frac{\overline{VM} \cdot \overline{VT}}{2} \cdot \frac{\overline{TK}}{3}$

$$= \frac{\overline{VM} \cdot \overline{VT} \cdot \overline{TK}}{6}.$$

The volume of the prism $QNH - V = \frac{\overline{MV} \cdot \overline{VT} \cdot VQ}{2}$.

And the volume

$$QNMV - KH = \overline{VM} \cdot \overline{VT} \cdot \left(\frac{VQ}{2} + \frac{TK}{6}\right)$$

Now since $\overline{HQ} = \overline{QN} = \overline{VT} = \overline{CW}$, and $\overline{VQ} = \overline{QG} = \overline{AW}$, if we denote $\overline{AC}$ by 2r and CW by h, the equation above becomes

Volume

$$XYZ - C = \pi ONMV - KH =$$

$$\pi \overline{VM} \cdot \overline{VT} \left(\frac{\overline{VQ}}{2} + \frac{\overline{TK}}{6}\right) = \pi h^2 \left(r - \frac{h}{3}\right)$$

Or:

The volume of a spherical segment equals the volume of a cylinder whose radius is the height of the segment and whose altitude is the radius of the sphere diminished by one-third the height of the segment.

THE PSYCHOLOGY OF ERRORS IN ALGEBRA

By PERCIVAL M. SYMONDS
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Teachers College, Columbia University

Ask a teacher why one of his pupils is doing poorly in algebra and you will receive one of three answers: that the pupil is stupid, or that he is careless, or that he is lazy. Whichever of these answers you receive is probably correct, but not very helpful. A teacher who thinks of his poorer pupils as stupid, careless, or lazy does not hold the key for remediation. If the boy is stupid, no remedy can cure him. Better teaching may make a little improvement in his accomplishment, increased motivation may lead to slightly greater achievement, but the net result will be meagre and unsatisfactory. Success in algebra does not come to the stupid boy—we find this evidenced in the high correlation which we find between algebraic ability and intelligence.

Likewise the teacher who diagnoses a cause of failure merely as carelessness, or as laziness, is not likely to do much in preventing such failure. The boy who fails in algebra does so because he does not do the examples. The boy who fails, fails because when confronted with an examination paper he has not become able to manipulate the problems so as to come out with the "correct answer" or even to reach any answer. Somewhere along the line he goes astray, he does the wrong thing—or is unable to do anything. In short he makes errors. Rugg and Clark in *Scientific Method in the Reconstruction of Ninth Grade Mathematics* (p. 81) divide errors into two groups: (1) accidental errors, and (2) recurring errors. By accidental errors these authors refer to errors of reading, writing, following directions, arithmetic, etc. Speaking precisely, these are no more accidental than any other kind of error, and after sufficient search, a satisfactory explanation can almost always be given in the individual case. "An accidental error is probably due to intermittent concentration, to lapses of attention, to instances of mind-wandering . . . one of the fundamental causes of failure in any school study is mental inertia and its accompanying weakness—mind-wandering, day-dreaming." The remedy for this is a more spirited class hour with little opportunity for this scattered at-

tention. The common practice of having examples already worked by the class put on the board and read aloud for the benefit of a minority who need to see the steps undoubtedly is a factor in building this habit of mind-wandering.

But more important for our present consideration are the recurring errors. By these the teacher means errors that a boy consistently makes. They are definitely formed habits, but wrong habits. Rugg and Clark have listed 44 such errors as they have found them to occur in their standardized tests. It is our purpose to show the psychology of these errors, and from this to point out methods of eradicating them, or better, of preventing the formation of these wrong habits.

Thorndike enumerates five characteristics of behavior. The first is multiple response to the same external situation. This is a grimly humorous truth in algebra. In how many ways may a boy respond to the situation $(x^3)^4$! He may write as an answer x^7, or x, or $x^{\frac{4}{3}}$, or x^{81}, or x^{12}. Or consider the problem, "If one part of x is a, what is the other part?" In putting this problem to a boy I received the following successive replies: "one," "it couldn't be," "ax," "xa." The teaching of algebra is made possible by the fact of multiple response on the part of the pupil. The problem of teaching algebra is: How can we ensure for ourself the one response that we desire?

A second characteristic of behavior is that the pupil has a set or determination in being presented with a problem—to get *the* answer. And when he has obtained *an* answer that in any way satisfies this set or determination, he is satisfied that he has *the* answer. There are three common satisfactions that determine in this way a pupil's choice of his answer. One is the social approval evidenced by the teacher or members of the class. I have found in watching boys work that for some the sole criterion for correctness is the nod or expression of the teacher. A favorable nod from the teacher is sufficient to convince this type of boy that he has not only found an answer that satisfies the teacher, but has found the correct answer, even though the answer really is wrong. A negative shake of the head will send the boy back again over his work to find his error. A perfectly neutral attitude on the part of the teacher will make the boy extremely uncomfortable, for it withholds from him his

only check on his result. Such a passive attitude will often evoke a question as, "Did I do it right?" Now this reliance on approval is not wholly bad. Most of our language habits are formed by approval of the correct form and disapproval of the incorrect form. It is only that our algebra is a more individual thing than language, and we expect, indeed, for success we must have, some other means by which the pupil may determine the correct answer independently.

A second "satisfaction" that determines a pupil's choice of his answer is an answer with which he may compare his—the answer book or key. The argument against keys is that the pupil will misuse them and will obtain the result in unfair ways. That is, we expect the pupil not only to get the answer, but to do the work of getting it, and usually in a prescribed way. But this objection is valid only where the work is defined as doing such and such exercises. Give a boy an assignment, "Pass in examples 1-12 on page 86 for tomorrow," and there is every incentive to copy or use answer books wrongly, but give a boy an assignment such as, "Practice these until you can get 12 right in 6 minutes," and the motive for improper use of the answer book is gone.

But somehow we expect our mathematicians to be more independent than this in determining correctness of answers. So a third and more intrinsic satisfaction is that of checking the answer. We all know these checks—after subtracting, adding the difference and bottom number to make the top number, multiplying the divisor and quotient to make the dividend, multiplying the factors to make the number to be factored, substituting the value of the unknown in the equation, substituting any number for the letters in a long and involved identity, etc. Almost all algebraic processes are either thus reversible, or may be checked by substituting. We sometimes think that checking is a troublesome and burdensome "extra" process which takes away time from the real drill at hand, but substituting the value of x in a literal equation often gives as difficult a process as the original solution. We should bear in mind that we *must* give the pupil some way of determining the correctness of his answer; if not teacher approval or keys, then we must allow the pupil the confidence that comes from checking.

Finally, with experience comes a divination of the fitness of an answer—as to the degree or dimensions of the answer, we have the feel as to whether the answer is .01 or 100.

Before leaving this topic we should consider one potent method a boy has of testing the correctness of his answer—namely, whether it comes out even or not. Early in his course, if it has not already been acquired in arithmetic, he becomes suspicious of any answer that does not come out even. In connection with the algebra inquiry, Mr. Orleans has made counts in nine texts in present use, to determine the number of answers that are integers, fractions, decimals, surds, or imaginaries. Following is a tabulation of his results expressed in percentages:

	A	B	C	D	E	F	G	H	I	Total
Integer	62.6	63.7	49.4	61.1	43.0	71.7	59.1	58.1	71.7	59.4
Fraction	15.7	19.7	14.6	15.6	13.4	23.4	14.2	15.1	10.6	15.4
Decimal	20.7	0.1	18.6	17.6	40.4	4.1	17.1	9.0	6.7	15.8
Surd	1.1	7.5	13.2	5.7	3.2	0.8	7.7	15.3	9.1	8.1
Imaginary	0.0	0.0	4.2	0.0	0.0	0.0	1.9	2.4	1.8	1.4

In any practical problem that a boy will ever meet, the chances are against "coming out even." Harvard College recognized this fact in its old entrance examinations and used to insert a quadratic equation with roots coming out "uneven." It is a bad habit to fix—that of judging the correctness of an answer by its "coming out even"—but no change can be made using our present texts unless teachers demand it.

While considering the second characteristic of behavior (*i. e.*, mental set or determination) one must not ignore the fact that sets other than that of obtaining the correct answer are present. Of special significance are those more enduring sets that a pupil has with regard to his algebraic ability as a whole. The set or attitude—"I can (or cannot) do algebra," "I do (or do not) like mathematics" and the like—are of great importance in determining the character of work in algebra. It is likely that nearly all the attitudes can be traced back to some signal occasion of success or failure. The boy that is suggestible to the extent of believing that $(x^3)^4 = x^7$ by receiving a favorable nod from his teacher is suggestible to the extent of believing that he is a failure in algebra if he receives "E" on his first quarter's report. In so far as our marking system is reliable, this as a good thing—it discourages the unfit from proceeding with the

to them impossible. But since our marking system is so grossly unreliable as we know it to be, this undoubtedly discourages many otherwise hopeful cases.

Some may object to the statement that the set of a pupil on being presented with a problem is "to get the answer"—they would like to have it "to pass the term's work," or "to learn algebra," or some more distant end. It is true that these more ideal or distant ends may be the more controlling attitude or determination of the more intelligent or sagacious pupils, but for the great majority the set is immediate and direct, to be satisfied immediately and directly. If the more distant aims are to be built up, they must be built by training as must any other habit.

A third characteristic of behavior is the partial response to a situation. This is a very common phenomenon in algebra—it partly is what prescribes a certain response instead of the multitude of others that might be made to any example. Usually it is clearer to think of it as the prepotency of certain elements of a situation. To take a concrete example, consider the familiar:

$$5x^2 + 16x + 3 = (5x + 3)\ (x + 1)$$

Here the boy obtains $5x$ and x to multiply to give $5x^2$; and 3 and 1 to multiply to give 3; and having obtained an answer, is satisfied. We must not necessarily convict the boy of forgetfulness. He is in much the same situation as the young teacher before his first class confronted by a room full of wide awake youngsters, and also by a subject matter which must be organized, and by the details of class room management such as distributing papers, attending to ventilation, keeping records, etc. Would we convict our young teacher of forgetfulness if he failed to attend to the ventilation in his early training? Rather he has expended his maximum attention on details of teaching. So our boy in the example above may have struggled so in getting his result that the fact of the cross products escaped him altogether. Some other examples are

$$a^2 + 10a + 24 = (a + 8)\ (a + 2)$$

where the prepotency of getting $8a + 2a = 10a$ has over-

shadowed getting something to multiply to 24. A still more obvious example is

$$12y + 18y^4 = 6(2y + 3y^4)$$

where only the numerical factors are taken out and the common literal factors ignored. This law of partial response works especially with signs:

$$-4(3x - 4) = -12x - 16$$

can be explained by the ignoring of the effect of the minus sign of the first 4 on the minus sign of the second 4. This is the more apparent when we come to the failure to change signs when the numerator of a fraction is preceded by the minus sign in such an example as

$$\frac{4x - 2}{3} - \frac{x - 3}{4} = 0$$

$$16x - 8 - 3x - 9 = 0$$

Here the effect of the sign of the fraction on the -3 is ignored. Other illustrations of this partial activity where attention is given to an element of the situation and other elements ignored are: $(ab^2)^2 = ab^4$, where the effect of the square on the a is ignored; $\sqrt{\frac{3}{5}} = \frac{1}{5}\sqrt{3}$ where the concentration is on getting rid of the denominator under the radical with consequent ignoring of the numerator: $\sqrt{x^5y^8} = x^4y^8\sqrt{x}$, where the concentration is on the factoring process with consequent ignoring of the square root process:

$$\frac{Mt - g}{t} = M - g$$

where the presence of a t in both numerator and denominator is the prepotent stimulus, and the fact that t is not distributed throughout the numerator is ignored. Finally,

$$-12x = 25$$

$$x = \frac{25}{12}$$

is an exasperating error that comes under this same category—

the attention is concentrated on the division, and the sign is neglected.

A fourth characteristic of behavior is what Thorndike calls "response by analogy." In any new situation one responds as he would respond to the situation most nearly resembling it in his past experience. Errors in algebra illustrate this. Consider the following:

$$x^2 - 81 = -0$$

$$x = 9$$

In linear equations, the result comes out $x =$ something, and the single value of x is a satisfactory result; consequently in quadratic equations the natural first response is to be satisfied with a single value of x. This error is also seen when a solution for a single letter is considered sufficient in solving simultaneous equations. Indeed, nearly all cases of unfinished examples can be traced back to a previous process where what is now a step in our unfinished example was then the final result. Even the answer $-x = 12$ can be so explained. Usually the pupil has his solution on dividing by the coefficient of x, i. e. $6x = 72$, $x = 12$, and consequently when he has divided by the coefficient 6 in $-6x = 72$, he has the cue that he has finished the example.

$$x^2 - 81 = 0$$

$$(x + 9)(x - 9) = 0$$

is a case of response by analogy—the usual response to $x^2 - 81$, combined with an ignoring of the

$$n^2 - 7n = 12$$

$$n(n - 7) = 12$$

is another case of the same nature.

$$V = Lwh$$

$$w = V - Lh$$

is a wrong application of the rule—"to get anything from one side of an equation to the other, change the sign."

Under this same consideration come those errors which are made unless a thing is taught specifically.

$$(a^2 + 3)\ (a^2 - 3) = a^2 - 9$$

a^2 is written in the answer because all of the previous examples of product and difference of two numbers have involved only numbers of the first power.

$$(3 \cdot 2)^2$$
$$= 3 \cdot 4$$
$$= 12$$

comes because the previous response to 2^2 has been 4.

$$x^7 \cdot x = x^7$$

comes because the previous connections have all been—to add visible exponents.

This analysis of errors makes prominent the fact that learning algebra is an extremely complicated process. No wonder it often seems like juggling letters. When a definite response has been made to a combination of letters and symbols, we may expect this response to persist until a definite effort towards change is made in it. Oftentimes these errors show how well previous processes have been learned. They surely show us how we have failed in emphasizing those elements of a problem that require a new or different response.

A fifth characteristic of behavior is associative shifting, the learning process itself. Out of the number of various responses that a pupil may make when confronted with a given problem, that one is selected which gives its satisfaction in the form of teacher approval, or correspondence with the answer in the answer book, or verification by checking, or fitting in with other known algebraic and quantitative facts. And by drill and repetition, these correct responses become definitely established habits.

A sixth characteristic of behavior—perseveration—is extremely important to explain certain of the so-called "accidental" errors. By perseveration is meant "a tendency to repeat an act time after time, when once it has been aroused by some appropriate stimulus . . . Perseveration is most apt to

lead to recall of recent experiences, and is most apt to appear when attention is relaxed, and the mind allowed to run freely. Instances of perseveration are found in the running of a tune in the head, soon after it has been heard; or in the flashing of a scene before the mind's eye soon after it has been actually seen; or in the reminiscences of the day which are apt to come to mind as one is dropping off to sleep." This perseveration is apt to occur in algebra whenever the pupil is not working at top notch enthusiasm, or when he is disturbed in any way by other trains of thought, be they relating to outside activities or to what is going on in the classroom besides the problem at hand. Occasionally, however, it may result from a too great concentration upon a particular feature of a problem. Arithmetical errors like $24 \cdot 8 = 182$ are undoubtedly explainable in this way—$8 \times 4 = 32$, and the 2 perseverates to the exclusion of the 3 and is combined with the $2 \times 8 = 16$, thus: $16 + 2 = 18$. Errors in writing, such as

$$(2x - 3)^2 = 4y^2 - 12x + 9$$

may come from perseveration from some previous example.

$$-4(3x - 4)$$
$$= -12x - 16$$

may be a case of the perseveration of the minus sign as well as a case of partial activity.

$$y^2 + y = 6$$
$$y = -3 \text{ or } -2$$

In cases where a boy does all of the work mentally, as in the above example, some element is quite apt to be perseverated, especially signs. In the above case, it may have been the minus sign before the 6 which he has to imagine when the 6 is transposed, or it may have been the minus sign of the 3 which he has just written. Such cases as

$$\sqrt{32} = 4\sqrt{4} \text{ and } \sqrt{50} = 5\sqrt{5}$$

rather than being errors in factoring, as Rugg and Clark suggest, point to a perseveration of the numerical value of the square root upon which he is concentrating. The boy does not write 16×2, but holds the confused picture in his mind, and in the confusion the 2 is lost.

I believe a seventh characteristic of behavior can be found which we may call *anticipation*. By this we mean the focussing of the attention on some element ahead of the writing in the process, which causes the reaction to occur before it should. This is the familiar error in speech when one says "prace rejudice" for "race prejudice," or "the stars have carted" for "the cars have started," or "fight the lires" for "light the fires." This comes usually at a time of special mental stress or mental eagerness—but as in perseveration, the attention is diverted from the mechanical reaction of speech in the present. It is hard to point out actual errors in algebra due to this cause—they are always "accidental" errors. Errors of reading, writing, and arithmetic may come under this heading. This phenomenon undoubtedly is a large factor in causing careless errors in examinations. For though the eagerness usually felt in an examination could not be called day-dreaming or mind-wandering, it might divert the attention from the specific process at hand.

The outcome of this psychological analysis of errors in algebra is that it enables us to improve our teaching in a way that epithets of "stupid," "careless," or "lazy" can never do. In the first place, it shows us the need for a psychological analysis of algebraic processes into the constituent connections or bonds involved. Especially those bonds that require a different response than in previous processes need analysis in order to receive emphasis in future teaching and drill. x^2x^7 without teaching would perhaps receive the answer x^{14}, because the customary thing to do when multiplying is to multiply the numbers on the scene. To add these exponents and obtain x^9 is an entirely new response in connection with numbers used in a multiplication process.

In the second place, our analysis of errors shows the need for drill or practice in various processes. Telling a boy what things to do does not form habits of doing them. The teacher ought not to expect that his explanation of a process is going to guarantee that the process is learned or a habit formed until the pupil has had a certain amount of drill on the process, not in fact until a test shows a certain proficiency in that process. Drill or practice is the remedy for attention to certain ele-

ments of an example to the exclusion of certain other elements. Not until certain processes become automatic can we expect the attention to be free to take in all the different steps needed in a solution. So the boy who omits the minus sign in solving $7x = -25$ needs more drill on short division so that the division will come automatically and attention can also take in the minus sign.

Whether we ought to attack a process like addition of fractions by learning many specific elements of the process, gradually combining and building up, or whether we should plunge the pupil at once into the final process, is still in doubt. It has been demonstrated that in learning poetry, nonsense syllables, etc., the whole method is superior to the part method. Poetry is a set of serial bonds, and usually there are no interferences from past connections to contend with. But algebra not only means the building up of a series of bonds, but bonds, some of which have facilitations from processes previously learned, and some of which have inhibitions from processes previously learned. There must be differential drill, with more emphasis on those connections that are hindered by past responses, and relatively less emphasis on those connections that are helped by past responses. The wise teacher will provide special drills of this propaedeutic nature.

Since much drill is necessary, and since only correct responses help build up correct habits, one should provide drill material that actually induces repetition of correct associations. I am referring to the custom of giving pupils a few difficult examples to do instead of many easy ones.

$$\left\{ \sqrt[5]{\frac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}} \times \frac{a\sqrt{x}}{x^{-1}\sqrt{a}} \right\}^{-4}$$

Instead of giving 5 problems comparable to the needless one above, where the pupil is apt to stumble about in the maze of exponents and radicals, it is better first to make sure that he has acquired a certain degree of proficiency with easy processes such as

$$x^{-2} = \frac{1}{x^2} \quad \text{and} \quad \sqrt{x} = x^{\frac{1}{2}}, \text{ etc.}$$

In the third place, since accidental errors are largely due to mind wandering, our psychological analysis shows the need for a reorganization of our classroom procedure. We need to develop, as Rugg and Clark say, a "spirited atmosphere" in the classroom. This is possible by (1) timing all formal activities. Work should be done under pressure. But to complement this Rugg and Clark show that we need (2) rapid shifts in the type of mental process involved. Instead of dragging through the whole hour on the involved difficulties of one topic it is better to vary the work, giving part of the time to rapid review drill, and part to the taking up of some new topic.

In closing, let me say a word in regard to the criticism that will undoubtedly be raised with regard to this psychological analysis of errors: that no allowance has been made for the reasoning capacity of the pupil. A child who makes the mistake $-4(3x-4) = -12x-16$ is stupid or careless, they say, because the function of the parenthesis is to distribute operations to everything within it. And so all errors offend the teacher's sense of the meaning of algebraic symbols. But let us not forget that we who teach mathematics know the very things that our pupils do not know. We see these mistakes to be inconsistent, because we have learned the meanings and uses of the symbols and see their interrelations—they do not see them to be errors because they have not learned the meanings of the symbols and do not comprehend their significance. Meaning comes with experience, and things take a significance after we have become adjusted to them. We as teachers have been assuming that our pupils acquire these meanings and relationships and apply them in their processes and manipulations. Psychology teaches us that the opposite is really the truth: our pupils first learn to carry out the processes and manipulations, and thereby acquire the meaning and significance of what they do.

PRESCRIBED VERSUS ELECTIVE MATHEMATICS IN JUNIOR HIGH SCHOOLS*

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1. Assume the existence of a large (1,600-pupil) junior high school, well staffed and amply supported, receiving all seventh and eighth grade pupils, and also all retarded pupils over 12 years of age. Assume that such a school makes no attempt to give specific training for any vocation or group of related vocations, but offers "vocationally flavored" courses, if thereby it can better motivate specific branches of the cultural, civic, and physical education (the components of general or liberal education) which are its primary concern.

2. It is possible, practicable, and desirable for such a school to offer a variety of one or two-year courses in mathematics, as for example:

a. A "stiff" two years' course in advanced arithmetic along lines of seventh and eighth grade work still traditional in many schools and textbooks.

b. One-year courses respectively in fairly "pure" algebra and plane geometry, slightly simplified from current high school standards in these subjects.

c. A one or two years' course in "composite" mathematics, incorporating the more vital and significant phases of arithmetic, algebra, geometry, etc., as suited to selected groups of the learners concerned.

d. A one or two years' course in applications of arithmetic and other mathematics subjects to the broad field of homemaking.

e. A one or two years' course in applications of mathematics to the more common fields of work in mechanical shops.

f. A one or two years' course in applications of arithmetic (and possibly samples from other fields) to the more common commercial occupations and transactions.

g. A one year's course in constructive geometry.

* Digest of an address before the New York Section of the Association of Teachers of Mathematics in the Middle States and Maryland, December 2, 1921.

h. A half course each year or grade devoted to reviewing, and giving new practical application to the "consumers" arithmetic learned by all normal pupils during the first six grades.

i. A one year's course in "advanced consumers' arithmetic" (beyond sixth grade standards).

j. Theoretically, a one year's course in "appreciational" mathematics corresponding to appreciational courses in music, poetry, graphic arts and general science.

3. All of the courses suggested above are "good" for society. Each one is "good," perhaps very good, for some class or type of junior school pupils. Some of them would certainly be of very little profit for certain other types of pupils. Possibly it would be educationally wicked to try to force some types of pupils to take certain of these courses.

4. Our assumed school contains some pupils of very low, some of very high, and many of modal abilities, general or even mathematical. It contains some from very poor environments, some from excellent environments, many from "fair American" environments. Some of its pupils will go through college, some will leave within two months after their 14th birthday, many will spend a portion of their time between the ages of 14 and 17 in some form of school, but with diminished purposiveness and educational effect.

5. The actual "case groups" which are the concern of educators who would, first, devise best possible curricula to meet "varying needs" (individual and societal) and, second, so administer these curricula as best to serve these needs, "cut across" the above classifications. For example, of pupils of low abilities some come from poor environments and may be expected to leave school early, whilst some come from prosperous environments and may be expected to attend school many years. Some pupils from poor homes will have very keen minds and strong working dispositions, but under present economic conditions they can have no serious expectations of even finishing high school, to say nothing of entering higher institutions. A large proportion of the best minds from the good environments will pass through liberal arts and professional colleges.

6. In view of the varieties of "case groups" in the school and the varieties of courses practicable to be offered, the school authorities are confronted by these problems:

a. Should *all* pupils be required to take in common certain work in mathematics? Why? If so, what?

b. Should *all* be required to take specified amounts of mathematics, choosing, under advice, from the various courses offered?

c. Should *all* pupils be permitted to elect, with advice of teachers and approval of parents, *much*, *little*, or *no* work in mathematics?

d. Should all be permitted to elect to take, or not to take, courses, subject to the condition that for admission to subsequent higher general, or specific vocational schools specified courses in mathematics will be required?

e. Shall certain pupils under specified conditions be prohibited from entering upon certain courses?

7. Contrary to much superficial educational opinion, the solutions of these problems do *not* rest alone upon assumptions or facts as to the *educational values* of the various mathematics courses for various typical groups of learners. The solutions are to be reached only through study of the *comparative* educational values of the many subjects now known to be available for junior high school curricula, the mastery of the total number of which would require several times the available time and learning energy of any pupil, even a brilliant one.

Each one of the courses suggested above is good, and each would, possibly, be good for all the pupils, if competing courses of greater value in other fields were not available. In actual curriculum making it will increasingly be a problem, not whether an advanced course in arithmetic, or a course in algebra, is "good" or "important" for a stated group of learners, but rather is it *as good*, or *as important*, as the modern language, music, English grammar, oral composition, general science, physical training, current events, civics, world history, drawing, industrial arts, household arts, geography, hygiene, general literature, vocational guidance or scouting, one or more of which must be omitted if such mathematical course is imposed or elected?

8. Who shall decide as to *relative educational values* of subjects available in given types of schools? Educational administrators have allowed specialist advocates with their *ex parte* pleadings, to play too large a part in determining curricula. The results are jumbles of compromises, giving the balance usually in favor of the strongest advocate, or else resulting in hopelessly congested curricula. A mathematician, like a musician, a geographer, a historian, a graphic artist, a business man, a home economics specialist, or a sociologist, is an expert in the content and technique of his field, and *sometimes* he is an expert in its *educational values* as a subject of study for stated groups of learners. But as an expert in the *relative* or *comparative* educational values of the various subjects that should, for stated groups of learners make up a *balanced school diet*—is he not as poorly equipped as any layman? What does the typical mathematician think of the "claims" of the musician, as to the educational values of his subject? And how does the latter reciprocate? And what does the geographer think of both of them? And how does the typical "English" specialist regard all three?

9. The educational authorities presupposed above finally establish a "jury" on educational values to advise them. This jury is composed of a superintendent of schools, a man with long experience as a junior high school principal, a social economist who has given much attention to the conditions of juvenile workers, an all-around business man, and a high school principal. That jury, after hearing various specialists in subject matter fields, a couple of psychologists, and some one who has made special studies of the careers and conditions of various classes of pupils after leaving school (cultural, civic, physical, and vocational), would, in the estimation of the present writer, endorse these solutions of the problems stated above:

a. No uniform course in mathematics should be prescribed in *common* for all pupils in seventh or eighth grades.

b. Too little is as yet known of the relative values of mathematics to justify us in requiring all pupils to take some course or courses. Hence, if pupils of certain classes desire, with the approval of their parents, to omit all mathematical studies during one or both of these grades, they should be permitted to do

so, provided they give evidence from time to time that the *consumers* arithmetic learned in the first six grades remains vitally functional, and provided that they pursue a full curriculum of other important courses.

c. Too little is yet known of the relative values of subjects, other than mathematics, to justify us in barring pupils from mathematical courses, except where a certain course is purposely designed only for learners of specified high grades of ability which the candidates in question do not possess.

d. As far as the imperfect powers of prophecy found among *ex officio* advisers permit, seventh and eighth grade pupils and their parents should be urged to give from ten to fifteen per cent of their school time to those mathematical courses, which, coming within their present probable abilities, are supposedly most nearly related to their probable future (not present) needs and interests. (But educators are under obligations to substitute knowledge for aspiration and belief as to what, for given vocations, higher studies, and consumers, actually are those future needs, differentiated somewhat perhaps among several ability or sumptuary levels.)

CERTAIN CASES OF EXTRANEOUS ROOTS*

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Sir Oliver Lodge has somewhere referred to the equation as the "most serious thing in mathematics." Certainly the solution of equations involves many serious difficulties in the teaching of mathematics. One such difficulty is the matter of so-called "extraneous roots."

An "extraneous root" may be defined as "a value obtained for an unknown in the solution of an equation which is not a root of the equation." It may be noted first that authors of textbooks usually fail to give a definition of extraneous roots; second, that the definition given above is faulty, in that it leaves out the qualifying phrase "by correct processes."

What constitutes "correct processes" in the solution of equations—in particular, in the solution of an equation in one unknown quantity?

The processes which, as a rule, are accepted as correct are:

(a) Adding equals to equals.
(b) Subtracting equals from equals.
(c) Multiplying equals by equals.
(d) Dividing equals by equals.
(e) Extracting equal roots of equals.

These processes are usually called "axiomatic" after the usage of Euclid in geometry. But it should be noted at once that Euclid nowhere makes any reference to anything resembling the last two processes above mentioned (for the obvious reason that in dealing with lines, angles, and areas, he treats them as if they were independent of number) and also that algebraic solution and geometric demonstration are not the same thing. In particular is it to be noted that the logic of an algebraic solution should hold when the steps are taken in reverse sequence, and that this is not necessarily true of a geometric demonstration.

This sets a certain limit upon the process of multiplying equals by equals. It is perfectly clear that dividing equals by

* Papers read before the Association of Mathematics Teachers of New Jersey, November 1, 1919, and October 30, 1920.

equals is not a valid process if the divisor is zero. It follows that multiplying equals by equals is not a valid process in the algebraic (reversible) argument, when the multiplier is zero.

For example, let us take the false equation

$$x + 7 = x + 5$$

If both sides are multiplied by $x - 3$, we have

$$x^2 + 4x - 21 = x^2 + 2x - 15$$

whence

$$x = 3$$

which does not check the original equation, nor will any other value of x check it. The difficulty is explained by the fact that both sides of the false equation have been multiplied by zero.

Whereas, in the equation

$$x^2 - 5x = 7x - 35$$

if both members are divided by $x - 5$, we have

$$x = 7$$

which *does* check, in spite of the fact that both members have been divided by zero in the form $x - 5$, since $x = 5$ will also check the equation.

This consideration would seem to contravene the familiar process of "clearing of fractions," when the L. C. D. contains the unknown quantity. As a matter of fact equations are often found in textbooks which, when treated in this fashion, yield results which are not roots at all.

For example:

$$\frac{x+3}{x^2-1} + \frac{3-x}{x-x^2} + \frac{x+2}{x^2+x} = 0.''$$

Multiplying by the L. C. D. $x(x+1)(x-1)$ we obtain

$$x^2 + 3x + x^2 - 2x + 3 + x^2 + x - 2 = 0$$

$$3x^2 + 2x - 5 = 0$$

Factoring

$$(3x + 5)(x - 1) = 0$$

Whence

$$x = -\tfrac{5}{3} \text{ or } x = 1$$

The value $-\frac{5}{3}$ gives a satisfactory check, but the value 1 gives the following:

$$\frac{4}{0} + \frac{3}{0} + \frac{8}{2} = 0 \text{ or } \text{“} \infty + \infty + \frac{8}{2} = 0\text{”}$$

Whatever meaning may be given to ∞, it is not likely to apply in the case of this result. In the best circles it would be said that there is no division by zero at all possible.

Consider also the equation

$$\frac{3}{2x+1} = \frac{4}{4x^2-1} + \frac{2}{1-2x}$$

Multiplying by the L. C. D. $(2x+1)(2x-1)$ we obtain

$$6x - 3 = 4 - 4x - 2$$
$$10x = 5$$
$$x = \tfrac{1}{2}$$

Checking $\frac{8}{2} = \frac{4}{0} + \frac{3}{0}$ or $\text{“}\frac{8}{2} = \infty + \infty\text{”}$ (?)

This means that the equation as given has no root at all. Since every equation must have a root if it has any meaning at all, it follows that the example given is not an equation, but is a falsehood in the form of an equation.

The difficulty, of course, lies in the fact that in general it is impossible to tell, when “clearing an equation of fractions,” whether or not one is multiplying by zero, that is, by an expression which equals zero for the same value which satisfies the equation itself.

As a matter of fact, any rational fractional equation (meaning an equation in which the unknown appears in some denominator) can be reduced to the form $P/Q = R$ where P, Q and R are integral functions. In this case, $P = QR$ follows from the axioms only if Q does not equal zero; and any value of x which makes $Q = 0$ cannot be called a root.

This raises the question as to the exact meaning of such an equation as $P/Q = R$.

If regard be had for the notion that division is merely the inverse of multiplication, this equation would mean that $P = QR$, in which, if $Q = 0$, P must also equal 0, which may or may not afford a root according as this is or is not possible.

As for example in the second illustration above cited:

$$\frac{3}{2x+1} - \frac{4}{4x^2-1} - \frac{2}{1-2x} = 0$$

$$\frac{6x-3-4+4x+2}{(2x+1)(2x-1)} = 0$$

$$\frac{10x-5}{(2x+1)(2x-1)} = 0$$

$$\frac{5(2x-1)}{(2x+1)(2x-1)} = 0$$

$$\frac{5}{2x+1} = 0$$

which must mean $5 = 0(2x+1)$, a palpable absurdity.

So far, the assumption has been that the fraction P/Q is in its lowest terms as regards the unknown; if it is not, then the equation may be written

$$\frac{TU}{TV} = R$$

which, if interpreted to mean

$$TU = RTV$$

leads to $$T(U - RV) = 0$$

which has two or more sets of roots.

This resolves at once the difficulty arising from the form 0/0, unless $P/Q = R$ is understood to signify something other than the equation $P = QR$, and to have infinitely many roots by the simple device of transforming the fraction to "higher terms." This is, of course, absurd, from the point of view of general theory. That is, $P = QR$ and $PS = QRS$ are really different and not equivalent equations. Whence it follows that $P/Q = R$ and $PS/QS = R$ are likewise different equations.

The conclusion is briefly—that any fractional equation set for solution should be of such a character that in its resultant form $P/Q = R$, P/Q is irreducible, and that the "extraneous roots" so-called are not roots at all.

How should this affect the teaching of elementary algebra? I am not prepared to dogmatize about the matter, as teachers may

in many instances prefer to continue with less exact methods for the sake of brevity, and textbook writers, in many cases, sidestep the whole question by care in the selection of exercises.

Irrational Equations

It follows at once that the familiar process of squaring both sides of an equation is not permissible, for the reason that the algebraic "axiom" "square roots of equals are equal" is not true, and this would be necessary to justify the step, as has been seen, since the axioms of algebraic demonstration apply *in reverse sequence* to the solutions of an equation.

For example, consider the false equation

$$\sqrt{x^2 - 13} = 1 - x$$

(This equation is false for the reason that there is no value of x which will make both members of the equation positive real numbers, as is necessary.)

Squaring both sides, we have

$$x^2 - 13 = 1 - 2x + x^2$$

whence

$$x = 7$$

which does not check, of course.

Whereas extracting the square root of each member of a quadratic equation properly arranged affords a new equation which in turn affords a solution which *will* check, as is well known.

How, then, are radical equations to be solved?

Let it be noted at once that radical equations of the sort found in elementary textbooks, and on college entrance examination papers, are usually special in character, concocted for the confounding of the minds of young persons, and rarely found in advanced or applied mathematics. The typical form usually contains three quadratic radicals—two radicals only not constituting a sufficiently severe test for the aspiring neophyte. Occasionally four radicals appear, but this practice is regarded as dangerous.

Let us first consider a case with two— or even one radical.

$$\sqrt{x^2 + 3x - 3} = 3x - 4$$

Solving by the usual method of squaring both sides, we obtain

$$x^2 + 3x - 3 = 9x^2 - 24x + 16$$
$$8x^2 - 27x + 19 = 0$$
$$(8x - 19)(x - 1 = 0$$
$$x = 1, \text{ or } 2\tfrac{3}{8}$$

The value 1 satisfies the original equation but the value $2\frac{3}{8}$ does not.

The student naturally wants to be shown why a root obtained by processes which he has been taught to consider correct is not a root at all. He is usually told that it is an extraneous root. This explains nothing and he is just as puzzled as before.

Let us now consider a solution by the use of a rationalizing factor.

Writing the equation in the form

$$\sqrt{x^2 + 3x - 3} - (3x - 4) = 0,$$

the rationalizing factor for the left hand member then is

$$\sqrt{x^2 + 3x - 3} + (3x - 4)$$

The question now arises: Can this rationalizing factor equal zero? As it is not permissible to multiply both members of an equation by zero, it becomes necessary to consider the result obtained when we have equations derived from two different functions of x, $f(x)$ and $\Phi(x)$, and a third derived from their product $f(x) \cdot \Phi(x)$. It does not take elaborate argument to show that all the roots of $\Phi(x) = 0$ are roots of $\Phi(x) \cdot f(x) = 0$, and all the roots of $f(x) = 0$ are roots of $\Phi(x) \cdot f(x) = 0$. The question which remains unanswered is: How many roots, if any, has an *irrational* equation $\Phi(x) = 0$? I have thus far not found any general test for the number of roots of an irrational equation. But given such an equation $\Phi(x) = 0$, we may term a second equation $f(x) = 0$, which is so formed that $\Phi(x) \cdot f(x)$ is rational, an *auxiliary equation*, if $f(x) = 0$ has roots. If not, then $f(x)$ can be regarded merely as a multiplier of both aides of the original $\Phi(x) = 0$.

Therefore returning to our problem, if we multiply member for member, the original equation,

$$\sqrt{x^2 + 3x - 3} - (3x - 4) = 0$$

by a rationalizing factor in the form of the auxiliary equation

$$\sqrt{x^2+3x-3}+(3x-4)=0$$

we obtain
$$x^2+3x-3-9x^2+24x-16=0$$
$$-8x^2+27x-19=0$$
$$-1(8x-19)(x-1)=0$$
$$x=1 \text{ or } 2\tfrac{3}{8}$$

The value 1 is a root of the original equation while the value $2\frac{3}{8}$ is a root of the auxiliary equation. It is not at all difficult to explain the origin of the so-called extraneous root when the equation is solved in this way.

It may be interesting to note in passing that frequently both values are given in an answer book without a word of comment, while it sometimes happens that a note like the following is given: "Certain values obtained for x in a radical equation are not roots of the equation as it appears."

Let us take another example

$$\sqrt{3x+7}-x-3=0$$

The rationalizing factor for the left-hand member is

$$\sqrt{3x+7}+x+3$$

This is an expression which no value of x will make equal to zero. Therefore we multiply *both* sides of the equation by this rationalizing factor and obtain

$$3x+7-x^2-6x-9=0$$
$$x^2+3x+2=0$$
$$(x+1)(x+2)=0$$
$$x=-2 \text{ or } -1$$

Both the value -2 and the value -1 are roots of the original equation.

Let us now consider equations with more than one radical in them. For example

$$\sqrt{x+1}+1-\sqrt{2x}=0$$

A rationalizing factor for the left-hand member is

$$\sqrt{x+1}+1+\sqrt{2x}$$

As this expression cannot equal zero, we multiply both members by it, obtaining

$$x+1+2\sqrt{x+1}+1-2x=0$$

whence

$$2\sqrt{x+1}-x+2=0$$

Just as we would find it necessary to square a second time if we were squaring both sides, so we now need to use a second rationalizing factor: *i. e.*,

$$2\sqrt{x+1}+x-2$$

As this expression may equal zero, we multiply member for member, the equation,

$$2\sqrt{x+1}-x+2=0$$

by the auxiliary equation

$$2\sqrt{x+1}+x-2=0$$

obtaining

$$4x+4-x^2+4x-4=0$$

whence

$$-x^2+8x=0$$

$$x=0 \text{ or } 8$$

The value 8 is a root of the original equation and the value 0 is a root of the auxiliary equation used to afford a second rationalizing factor.

Let us take another case with three radicals in the equation:

$$2\sqrt{x+1}+\sqrt{x-2}-\sqrt{7x+4}=0$$

A rationalizing factor for the left hand member is

$$2\sqrt{x+1}+\sqrt{x-2}+\sqrt{7x+4}$$

As this expression cannot equal zero, we multiply both members by it, obtaining

$$4x+4+4\sqrt{x^2-x-2}+x-2-7x-4=0$$

$$4\sqrt{x^2-x-2}-2x-2=0$$

$$2\sqrt{x^2-x-2}-x-1=0$$

The second rationalizing factor is

$$2\sqrt{x^2-x-2}+x+1$$

As this expression may apparently equal zero, we multiply, member for member, the equation

$$2\sqrt{x^2-x-2}-x-1=0$$

by the auxiliary equation

$$2\sqrt{x^2-x-2}+x+1=0$$

obtaining

$$\begin{aligned} 4x^2-4x-8-x^2-2x-1&=0\\ 3x^2-6x-9&=0\\ x^2-2x-3&=0\\ (x-3)(x+1)&=0\\ x&=3 \text{ or } -1 \end{aligned}$$

The value 3 is a root of the original equation and -1 is a root of the auxiliary equation used as the second rationalizing factor.

Let us take one more case—that of a radical equation in the quadratic form.

$$3x^2-4x+2+3\sqrt{3x^2-4x+2}-4=0$$

Factoring

$$(\sqrt{3x^2-4x+2}+4)(\sqrt{3x^2-4x+2}-1)=0$$

Multiplying by the auxiliary equation

$$(\sqrt{3x^2-4x+2}-4)(\sqrt{3x^2-4x+2}+1)=0$$

we obtain

$$(3x^2-4x+2-16)(3x^2-4x+2-1)=0$$

$$(3x^2-4x-14)(3x-1)(x-1)=0$$

whence

$$x=\frac{2\pm\sqrt{46}}{3},\ \tfrac{1}{3},\text{ or } 1$$

The values $\frac{1}{3}$ and 1 are roots of the original equation while $\frac{2\pm\sqrt{46}}{3}$ are roots of the auxiliary equation used to afford a rationalizing factor.

I could continue to give examples taken from standard text books, but I think a sufficient number has been given to illustrate my contention that the logic of an algebraic solution should hold when the steps are taken in reverse sequence.

ARITHMETIC IN THE HIGH SCHOOL

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The object of this paper is to call attention to some tendencies in the American High School which the writer considers dangerous to the coming generations of high school graduates. It is a plea for a more extended study of practical mathematics. In presenting his view the writer may seem critical and, at times, harsh and radical. He believes, however, that the times are harsh, and require more or less radical statements to cause a change in the present drift of the high school curriculum.

Up to a comparatively few years ago the object of all public education seemed to be to get students to read well and understandingly, to write legibly, and to figure accurately. Higher education, so-called, was left to the colleges and universities. Such a small percentage of the high school graduates could ever attend college that a demand was created for higher education in the local school. Hence the high school was developed. The early high school became a stepping stone to college—in other words, a preparatory school for further study in the greater schools. The course presented was, naturally, that which would make it possible for a graduate to enter at most any college without further preparation. It was not considered the function of the high school to fit its graduates for anything in particular—they were supposed to either enter college or go into the industries starting at the bottom along with the uneducated and working up. Many people believed that the high school should do more for the boy who could not attend college—that he could be given a more definite training in the public schools and be thereby fitted to start a little higher up on the industrial ladder of success. As a result, the high school added a large number of courses which we choose to call Vocational, the aim of each one being to fit the student for some definite line of work. The high school had then two aims for its entering students—they must either prepare for college or for a definite vocation. These two aims, both worthy of any good American High School, because of the multiplicity of the subjects neces-

sarily taught have caused endless conflict. It seems that in the struggle to meet the demands for college preparation, and at the same time to give the student something definite in case he decides to not go to college many educators have lost sight of some of the most fundamental things necessary for ultimate success.

Mr. William McChesney Martin, Chairman of the Board of the St. Louis Federal Reserve Bank, in answer to a question as to what is the most desired characteristic of a young man entering the business world today said, promptly and decisively, *initiative*. The ability to independently analyze a problem with accuracy even though it be an entirely new one—the ability to think clearly and logically—the stick-to-it-iveness that carries a person through a difficult solution of a business problem—all these are absolutely necessary to the successful man or woman of today.

Are these qualities born in a person or can they be acquired by training and education? Might the writer ask if a great athlete was just born that way—if a great musician was just the freak of birth—if any really pronounced success was just the result of chance? You who hear this paper will all agree that to every one who has become a pronounced success because of being born a genius there are a hundred who acquired only a limited ability by birth, but who, because of their training, became the leaders of today. As teachers of science and mathematics we are convinced that in the proper amount of the right kind of training lies the hope of the coming generations. Initiative can be acquired through training the same as we improve or acquire any of the God given gifts of nature.

We often confuse initiative with impulse. The writer has often heard it said of a person who was very impulsive that they had so much initiative. We more or less admire an impulsive person, but most impulsive people get in the way at times and are more of a detriment than otherwise. Impulse is the wrong brand of initiative. It is the natural born initiative untrained. Initiative of the right sort must come from the person who is able to think accurately, to reason down into the very heart of things, and then to act.

If initiative is a process of correct reasoning, then our schools should have courses designed to develop this faculty. To be able

to reason a person must be possessed of the necessary facts and be able to arrange these facts in a logical way to arrive at a valid conclusion. Possibly the best place to teach reasoning is in the science laboratory or in the mathematics classes. These branches deal with concrete facts. The youth of our land must be shown concrete things; not necessarily because his head is part concrete, but because this is an age of doubt and investigation. An instructor of a century ago was believed because his students thought he knew all there was to be known of the subject. So little was generally known of the subject taught that the student had small opportunity for proof. Nowadays the student accepts no statements of his instructor unless it can be proved by concrete facts. We must recognize this in the student of today and we must satisfy his mind if we are to teach him to reason at all. Because of the exact nature of the reasoning and the concrete materials shown it seems that the best results should be obtained from the courses mentioned—science and mathematics.

But—and here begins the sad part of this story—the crowding of so many different courses—the overcrowding of the curriculum—has made it impossible for all students graduating from high school to take a sufficient number of courses in either science or mathematics. The writer is particularly interested in the product of the high school from the commercial field and has made an intensive study of the problems of that course. In many high schools a student may graduate from the Commercial, and in fact from many other special courses with practically no training in mathematics. Those who are interested in these courses have demanded so many special types of training that the fundamental courses have been crowded out. The academic teachers, on the other hand, have done little to stem the tide. They have failed to add things in their courses that would make them more desirable from the standpoint of the vocational student. As a result, both the special course and the college course students have lost. This fact is proved beyond a doubt where a so-called General Course is offered—the majority of the students will graduate from the general course because it offers them possible college entrance requirements as well as some of the more practical subjects.

The writer can see no reason why the academic courses cannot be practical and at the same time be the best preparation for college or industry. The fault must lie in the subject matter or in the method of presentation.

In the early development of the Commercial course it was found necessary to teach a special course in Arithmetic. The work in mathematics in the graded schools could not go far enough to train the boy to handle many of the complex problems of the business world. In many high schools this course in Arithmetic was all the study in mathematics offered the graduate from the commercial course. This was radically wrong. The study of Arithmetic was necessarily a series of mental gymnastics to give the student the power to carry on the fundamental processes of calculation with accuracy and dispatch. It was a means to give the student immediate ability in figures. It did not teach the ability to reason to any great extent though this was all of this type of training the commercial student received.

On the other hand it was not thought necessary that a commercial student needed a course in Algebra. In fact, it is difficult for the commercial student to take courses in Algebra because of the complexity of the requirements of his course. Yet higher forms of Accounting demand a very extended use of the higher branches of mathematics. The difficulty in the past has been that the course in Algebra did not attract the Commercial student. The regular mathematics teacher will probably feel offended when the writer claims that much of the work now done in Algebra fails to produce results. The real teacher of Algebra seeks to lead his students into the realm of mathematical reason. He strives to prepare him to reason out the problems of life through the unknown quantity in the Algebraic equation. He seeks to reduce everything to equations, and thereby lay the foundation for more intensive study of mathematical problems. Unless he is an exceptional teacher he fails in the one most vital aim of teaching—that of getting his students to realize the connection of algebra with his possible life work. The student takes the course in Algebra because he has been advised that it is a necessary foundation for his career. In most cases he is, however, unable to trace any connection, either before or after, with his life work. The study of Algebra

has gone into so much disrepute that it is no longer required for graduation from many high schools.

The relegating of Algebra to the background is one of the most serious and dangerous tendencies of the high school today. It must be given its place of prominence once more. As teachers, we all realize what the study of mathematics means in the development of the future man, but it is hard to show parents when they are convinced that their boy can get along just as well without it. People to-day are looking for immediate results—they want something new. Unless a course in high school shows some real practical value now the parent is against it or at least indifferent. No matter how strongly we may feel that the substitution of anything for a strong course in mathematics is radically wrong the fact still remains that outside interests are demanding the substitution of other subjects for Algebra.

A proper solution of this problem seems to demand the offering of a course in General Mathematics covering a period of four terms work. The Algebra teachers are constantly facing the problem of students who are weak in the fundamental principles of mathematics—they cannot add, subtract, divide, or multiply. The present Algebra teacher would be much better off if the student were given a strong course in these fundamental principles before touching Algebra.

On the other hand a boy enters high school and finds himself in a different atmosphere; he is treated differently, is left to get his work by himself to a considerable extent. There is a great gap between the graded school and the high school in the method of instruction. Often the new student is so bewildered that he is a long time in finding himself. The large number of those taking Algebra who fail the first term is a good illustration of the point that we need some more mathematical instruction before delving into Algebra.

The number of students who leave high school is the greatest in the first year. What becomes of these students? They do not go to college—they go to work. Then we ought to give in the first year in high school those courses that make it possible for the large number who will be forced to leave school to get something definite and practical. Algebra will do them no particular good but Arithmetic will. The kind of Arithmetic given

should not be the same as that given in the grades, but it should be of such a character that it will have the same value as preparation for college as any other branch of mathematics.

What should we teach in this course in Arithmetic? Most teachers merely follow the book—"tomorrow we will take the next ten problems" is the assignment of the average teacher. The writer has never seen Arithmetic taught to suit him, but he believes it can be taught so that the student will see its intense practical application and also realize something of the romance of the workaday business world. A great deal has been written already about teaching mathematics objectively. If there is any subject that should be taught that way it is high school arithmetic.

In the first place, the Arithmetic course should be divided into a number of units varying in their subject matter in accordance with the needs of the community. If I were teaching in a distinctly farming community I would use those units having direct bearing upon farming and farm products. I would teach units which lay emphasis upon weights and measures—the mathematics of marketing—one treating of the problem of transportation—one on money and banking. Of course along with these units would be continuous drill on the fundamental principles of calculation. Any boy taking this term's work would be fitted to handle the necessary calculations in the farming community.

For the metropolitan district these units should be varied. Bring in a study of the factory payroll, the sale and purchase of merchandise from both the retail and the wholesale standpoint, billing, and a great many others. These units should of course be placed in the course in such a logical way that if the pupil dropped out at any time he would have something definite to depend upon for a job. In the great majority of cases the students who drop out of the school in the first year or two in the large cities either become clerical workers or factory hands. This training in Arithmetic would help them get a better job.

The student who wished to go on and finish his course would find Algebra much easier after the course in Arithmetic and he would be less likely to drop school because of failure in mathe-

matics. A bridge would thus have been thrown across the gap separating the graded school from the high school—no, the millenium would not have been reached, but the boy going out of our high school either a graduate or a "quituate" would be vastly better fitted to cope with the problems of life.

The study of Arithmetic in high school should not be limited to the Commercial student, but it should be a part of the fundamental study in mathematics so necessary to training the mind of the American Boy.